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Langevin equation in terms of conformable differential operators

Bashir AHMAD, Ravi P. AGARWAL, Madeaha ALGHANMI and Ahmed ALSAEDI

Abstract

In this paper, we establish sufficient criteria for the existence of solutions for a new kind of nonlinear Langevin equation involving conformable differential operators of different orders and equipped with integral boundary conditions. We apply the modern tools of functional analysis to derive the desired results for the problem at hand. Examples are constructed for the illustration of the obtained results.

1 Introduction

Langevin equation is an important tool of mathematical physics, which successfully describes the processes like anomalous diffusion, price index fluctuations [1], fractal environment in the irreversible dynamics of a harmonic oscillator [2] and so forth. When the separation of the microscopic and macroscopic time scales does not exist in the systems, the fractional analogue (also known as stochastic differential equation) of the usual Langevin equation is suggested, for example, see [1]. In [3], the author investigated moments, variances, position and velocity correlation for a fractional Langevin equation with Riemann-Liouville fractional time derivative and compared the obtained results with the ones derived for the same generalized Langevin equation involving Caputo fractional derivative. For some recent works on boundary

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value problems involving Langevin equation, we refer the reader to the papers [4] - [9] and the references cited therein.

In [10, 11], the concept of so called "conformable fractional differential and integral operators" was introduced and elaborated. However, the conformable fractional differential operator has no relation with fractional calculus. It is better to call such an operators as "conformable differential operator". In [12], some more results for conformable calculus were obtained. In [13], the authors studied the stability and asymptotic stability of conformable nonlinear differential systems by using Lyapunov function. In a recent article [14], the authors discussed the existence of positive solutions for a conformable differential equation equipped with integral boundary conditions.

In the present paper, we introduce a new type of nonlinear Langevin equation involving conformable differential operators of different orders and solve it with integral boundary conditions. In precise terms, we investigate the existence of solutions for the following problem:

$$\begin{cases} T_\alpha(T_\beta+\lambda)x(t)=f(t,x(t)),\ \lambda,\,\mu\in\mathbb{R},\ t\in J:=[0,\tau], \\ x(0)=0,\quad x(\tau)=\mu\int_0^\tau x(s)ds,\,\mu\in\mathbb{R}, \end{cases} \tag{1}$$

where T_α, T_β are the conformable differential operators of order $\alpha,\beta\in(0,1]$ and $f:[0,\tau]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

2 Preliminaries

In this section, we briefly describe some basic concepts of conformable calculus [10, 11], related to our work. In the sequel, we omit the word "fractional" from the related literature.

Definition 2.1 For $\alpha\in(0,1]$, the conformable derivative of a function $f:[a,\infty)\to\mathbb{R}$ of order α is defined by

$$T_\alpha^a f(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon(t-a)^{1-\alpha})-f(t)}{\epsilon},\text{ for all }\ t>a. \tag{2}$$

If $T_\alpha^a f(t)$ exist on (a,b) then $T_\alpha^a f(a)=\lim_{t\to a^+}T_\alpha^a f(t)$.

Definition 2.2 Let $\alpha\in(n,n+1]$. The conformable derivative of a function $f:[a,\infty)\to\mathbb{R}$ of order α when $f^{(n)}(t)$ exists, is defined by

$$T_\alpha^a f(t)=T_{\alpha-n}^a f^{(n)}(t). \tag{3}$$

Definition 2.3 Let $\alpha\in(n,n+1]$. The conformable integral of a function $f:[a,\infty)\to\mathbb{R}$ of order α is defined by

$$I_\alpha^a f(t)=\frac{1}{n!}\int_a^t (t-s)^n(s-a)^{\alpha-n-1}f(s)ds. \tag{4}$$

Lemma 2.1 Let $\alpha \in (n, n+1]$. If $f^{(n)}(t)$ is a continuous function on $[a, \infty)$, then, for all $t > a$, $T_\alpha^a I_\alpha^a f(t) = f(t)$.

Lemma 2.2 Let $\alpha \in (n, n+1]$. Then $T_\alpha^a (t-a)^k = 0$ for all $t \in [a, b]$ and $k = 1, 2, \ldots, n$.

Lemma 2.3 Let $\alpha \in (n, n+1]$. If $T_\alpha^a f(t)$ is a continuous function on $[a, \infty)$, then

$$I_\alpha^a T_\alpha^a f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}, \ \ \forall t > a. \tag{5}$$

In passing, we remark that I_α and T_α respectively denote the conformable integral and conformable derivative of f with $a = 0$.

Lemma 2.4 Let $h \in C(0, \tau)$. Then the unique solution of the boundary value problem:

$$\begin{cases} T_\alpha(T_\beta + \lambda)x(t) = h(t), \quad t \in J := [0, \tau], \\ x(0) = 0, \quad x(\tau) = \mu \int_0^\tau x(s)ds, \end{cases} \tag{6}$$

is given by

$$\begin{aligned} x(t) = \quad & I_\beta(I_\alpha h(s))(t) - \lambda I_\beta x(t) + \frac{t^\beta}{\beta\Omega}\Big\{\mu \int_0^\tau I_\beta(I_\alpha h(u))(s)ds - I_\beta(I_\alpha h(s))(\tau) \\ & -\lambda\Big(\mu \int_0^\tau I_\beta x(s)ds - I_\beta x(\tau)\Big)\Big\}, \end{aligned} \tag{7}$$

where it is assumed that

$$\Omega := \frac{\tau^\beta(\beta + 1 - \mu\tau)}{\beta(\beta+1)} \neq 0. \tag{8}$$

Proof. Applying the operator I_α on both sides of the fractional differential equation in (6) and using Lemma 2, we obtain

$$(T_\beta + \lambda)x(t) = I_\alpha h(t) + c_1, \tag{9}$$

for some $c_1 \in \mathbb{R}$. Next, applying the operator I_β on both sides of (9), we get

$$x(t) = I_\beta(I_\alpha h(s))(t) - \lambda I_\beta x(t) + c_1 \frac{t^\beta}{\beta} + c_2. \tag{10}$$

Using the boundary conditions given by (6) in (10), we find that $c_2 = 0$ and

$$\begin{aligned} c_1 \quad = \quad & \frac{1}{\Omega}\Big\{\mu \int_0^\tau I_\beta(I_\alpha h(u))(s)ds - I_\beta(I_\alpha h(s))(\tau) \\ & -\lambda\Big(\mu \int_0^\tau I_\beta x(s)ds - I_\beta x(\tau)\Big)\Big\}. \end{aligned}$$

Substituting the values of c_1 and c_2 in (10) yields the solution (7). The converse follows by direct computation. The proof is completed. $\square$

3 Existence and uniqueness results

In this section, we derive existence and uniqueness results for the problem (1) in a Banach space $\mathcal{C}$ of all continuous functions from $[0,\tau]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup_{t\in[0,\tau]} |x(t)|$. By Lemma 2, we define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \mathcal{G}(x)(t) \quad = \quad & I_\beta(I_\alpha f(s,x(s)))(t) - \lambda I_\beta x(t) + \frac{t^\beta}{\beta\Omega}\Big\{\mu\int_0^\tau I_\beta(I_\alpha f(u,x(u)))(s)ds \\ & -I_\beta(I_\alpha f(s,x(s)))(\tau) - \lambda\Big(\mu\int_0^\tau I_\beta x(s)ds - I_\beta x(\tau)\Big)\Big\}. \end{aligned} \tag{11}$$

For brevity, we set the notations:

$$\Lambda_1 = \frac{\tau^{\alpha+\beta}}{\alpha(\alpha+\beta)}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{(\alpha+\beta+1)}\Big)\Big\}, \tag{12}$$

and

$$\Lambda_2 = |\lambda|\frac{\tau^\beta}{\beta}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{\beta+1}\Big)\Big\}. \tag{13}$$

Our first result dealing with the existence of solutions for the problem (1) relies on Krasnoselskii's fixed point theorem [15], which is stated below.

Lemma 3.1 (Krasnoselskii's fixed point theorem). Let E be a Banach space and $\mathcal{U}$ be a closed convex and nonempty subset of E. Let $\mathcal{A}_1, \mathcal{A}_2$ be the operators defined on $\mathcal{U}$ to E such that (i) $\mathcal{A}_1 u + \mathcal{A}_2 v \in \mathcal{U}$ whenever $u, v \in \mathcal{U}$; (ii) $\mathcal{A}_1$ is compact and continuous; and (iii) $\mathcal{A}_2$ is a contraction. Then there exists $s \in \mathcal{U}$ such that $s = \mathcal{A}_1 s + \mathcal{A}_2 s$.

Theorem 3.2 Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition:

(H_1) there exists a continuous function $\psi \in C([0,\tau], \mathbb{R}^+)$ such that

$$|f(t,x)| \le \psi(t), \quad \forall (t,x) \in J \times \mathbb{R}.$$

Then the problem (1) has at least one solution on J, provided that

$$\Lambda_2 < 1, \tag{14}$$

where Λ_2 is given by (13).

Proof. Consider the set $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ with $r > \frac{\|\psi\|\Lambda_1}{1-\Lambda_2}$, where $\|\psi\| = \sup_{t\in[0,\tau]} |\psi(t)|$, Λ_1 and Λ_2 are given by (12) and (13) respectively. Define operators $\mathcal{G}_1$ and $\mathcal{G}_2$ from B_r to $\mathcal{C}$ as follows:

$$\begin{aligned}\mathcal{G}_1(t) &= I_\beta(I_\alpha f(s,x(s)))(t)\\ &\quad +\frac{t^\beta}{\beta\Omega}\Big\{\mu\int_0^\tau I_\beta(I_\alpha f(u,x(u)))(s)ds - I_\beta(I_\alpha f(s,x(s)))(\tau)\Big\},\\ \mathcal{G}_2(t) &= -\lambda I_\beta x(t) - \frac{\lambda t^\beta}{\beta\Omega}\Big\{\mu\int_0^\tau I_\beta x(s)ds - I_\beta x(\tau)\Big)\Big\}.\end{aligned}$$

Notice that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ on B_r. Now we verify the hypothesis of Lemma 3. For $x, y \in B_r$, we find that

$$\begin{aligned}&\|\mathcal{G}_1 x + \mathcal{G}_2 y\|\\ \leq\ & \sup_{t\in J}\Bigg\{I_\beta(I_\alpha|f(s,x(s))|)(t) + |\lambda| I_\beta|y(t)| + \frac{t^\beta}{\beta|\Omega|}\Big\{|\mu|\int_0^\tau I_\beta(I_\alpha|f(u,x(u))|)(s)ds\\ &+I_\beta(I_\alpha|f(s,x(s))|)(\tau) + |\lambda|\Big(|\mu|\int_0^\tau I_\beta|y(s)|ds + I_\beta|y(\tau)|\Big)\Big\}\\ \leq\ & \|\psi\|\Bigg\{\frac{\tau^{\alpha+\beta}}{\alpha(\alpha+\beta)}\Big\{1+\frac{\tau^\beta}{\beta|\Omega|}\Big(1+|\mu|\frac{\tau}{(\alpha+\beta+1)}\Big)\Big\}\Bigg\}\\ &+\|y\|\Bigg\{|\lambda|\frac{\tau^\beta}{\beta}\Big\{1+\frac{\tau^\beta}{\beta|\Omega|}\Big(1+|\mu|\frac{\tau}{\beta+1}\Big)\Big\}\Bigg\}\\ \leq\ & \|\psi\|\Lambda_1 + r\Lambda_2 < r,\end{aligned}$$

which implies that $\mathcal{G}_1 x + \mathcal{G}_2 y \in B_r$. Next, we show that $\mathcal{G}_2$ is a contraction. For that, let $x, y \in \mathcal{C}$. Then

$$\begin{aligned}\|\mathcal{G}_2 x - \mathcal{G}_2 y\| &\leq \sup_{t\in J}\Bigg\{|\lambda| I_\beta|x(t)-y(t)| + |\lambda|\frac{t^\beta}{\beta|\Omega|}\Big\{|\mu|\int_0^\tau I_\beta|x(s)-y(s)|ds\\ &\quad + I_\beta|x(\tau)-y(\tau)|\Big\}\Bigg\}\\ &\leq |\lambda|\frac{\tau^\beta}{\beta}\Big\{1+\frac{\tau^\beta}{\beta|\Omega|}\Big(1+|\mu|\frac{\tau}{\beta+1}\Big)\Big\}\|x-y\|\\ &= \Lambda_2\|x-y\|,\end{aligned}$$

which together with the condition (14) implies that $\mathcal{G}_2$ is a contraction. Continuity of f implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on B_r as

$$\|\mathcal{G}_1 x\| \leq \|\psi\|\Lambda_1.$$

In order to show the compactness of the operator $\mathcal{G}_1$, let $\sup_{(t,x)\in J\times B_r}|f(t,x)| = \bar{f} < \infty$. Then, for $t_1, t_2 \in J$, $t_1 < t_2$, we have

$$\begin{aligned} & |(\mathcal{G}_1x)(t_2) - (\mathcal{G}_1x)(t_1)| \\ \leq & \Big| \int_0^{t_2} s^{\beta-1}\Big(\int_0^s u^{\alpha-1} f(u,x(u))du\Big)ds - \int_0^{t_1} s^{\beta-1}\Big(\int_0^s u^{\alpha-1} f(u,x(u))du\Big)ds \\ & + \frac{t_2^\beta - t_1^\rho}{\beta\Omega}\Big\{\mu\int_0^\tau I_\beta(I_\alpha f(u,x(u)))(s)ds - I_\beta(I_\alpha f(s,x(s)))(\tau)\Big\}\Big| \\ \leq & \Big(\frac{\bar{f}|t_2^{\alpha+\beta} - t_1^{\alpha+\beta}|}{\alpha(\alpha+\beta)} + \frac{\bar{f}|t_2^\beta - t_1^\rho|}{\beta|\Omega|}\Big\{\frac{\tau^{\alpha+\beta}}{\alpha(\alpha+\beta)}\Big(1 + |\mu|\frac{\tau}{(\alpha+\beta+1)}\Big)\Big\}\Big) \to 0 \\ & \text{as } t_2 \to t_1, \end{aligned}$$

independently of $x \in B_r$. Thus $\mathcal{G}_1$ is equicontinuous. So $\mathcal{G}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_1$ is compact on B_r. Thus all the assumptions of Lemma 3 are satisfied. So the conclusion of Lemma 3 applies and that the problem (1) has at least one solution on J. □

In the next result, we prove the uniqueness of solutions for the problem (1) by applying Banach's fixed point theorem.

Theorem 3.3 Assume that $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following condition:

(H_2) there exists a positive constant $\mathcal{L}$ such that

$$|f(t,x) - f(t,y)| \leq \mathcal{L}|x-y|, \quad \text{for } t \in J \text{ and for every } x, y \in \mathbb{R}.$$

Then there exists a unique solution for the problem (1) on J provided that

$$\mathcal{L}\Lambda_1 + \Lambda_2 < 1, \tag{15}$$

where Λ_1 and Λ_2 are respectively given by (12) and (13).

Proof: Let us define $B_{\bar{r}} = \{x \in C([0,T],\mathbb{R}) : \|x\| \leq \bar{r}\}$ with

$$\bar{r} \geq \frac{\Lambda_1 M}{1 - L\Lambda_1 - \Lambda_2}, \ \sup_{t\in[0,\tau]}|f(t,0)| = M$$

and show that $\mathcal{G}B_{\bar{r}} \subset B_{\bar{r}}$, where the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by (11).

For $x \in B_{\bar{r}}$, using (H_2), we get

$$
\begin{aligned}
& |\mathcal{G}(x)(t)| \\
\leq\ & I_\beta(I_\alpha[|f(s,x(s)) - f(s,0)| + |f(s,0)|])(t) + |\lambda| I_\beta |x(t)| \\
& + \frac{t^\beta}{\beta|\Omega|}\Big\{|\mu| \int_0^\tau I_\beta(I_\alpha[|f(u,x(u)) - f(u,0)| + |f(u,0)|])(s)ds \\
& + I_\beta(I_\alpha[|f(s,x(s)) - f(s,0)| + |f(s,0)|])(\tau) + |\lambda|\Big(|\mu| \int_0^\tau I_\beta|x(s)|ds + I_\beta|x(\tau)|\Big) \\
\leq\ & (\mathcal{L}\bar{r} + M)\Big(\frac{\tau^{\alpha+\beta}}{\alpha(\alpha+\beta)}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{(\alpha+\beta+1)}\Big)\Big\}\Big) \\
& + \bar{r}\Big(|\lambda|\frac{\tau^\beta}{\beta}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{\beta+1}\Big)\Big\}\Big) \\
=\ & (\mathcal{L}\bar{r} + M)\Lambda_1 + \Lambda_2\bar{r} \leq \bar{r},
\end{aligned}
$$

which, on taking the norm for $t \in [0,\tau]$, yields $\|\mathcal{G}(x)\| \leq \bar{r}$. This shows that $\mathcal{G}$ maps $B_{\bar{r}}$ into itself. In order to show that the operator $\mathcal{G}$ is a contraction, let $x, y \in C([0,\tau],\mathbb{R})$. Then, for each $t \in [0,\tau]$, we obtain

$$
\begin{aligned}
|\mathcal{G}(x)(t) - \mathcal{G}(y)(t)| =\ & I_\beta(I_\alpha|f(s,x(s)) - f(s,y(s))|)(t) + |\lambda| I_\beta|x(t) - y(t)| \\
& + \frac{t^\beta}{\beta|\Omega|}\Big\{|\mu|\int_0^\tau I_\beta(I_\alpha|f(u,x(u)) - f(u,y(u))|)(s)ds \\
& + I_\beta(I_\alpha|f(s,x(s)) - f(s,y(s))|)(\tau) \\
& + |\lambda|\Big(|\mu|\int_0^\tau I_\beta|x(s) - y(s)|ds + I_\beta|x(\tau) - y(\tau)|\Big) \\
\leq\ & \mathcal{L}\|x - y\|\Big(\frac{\tau^{\alpha+\beta}}{\alpha(\alpha+\beta)}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{(\alpha+\beta+1)}\Big)\Big\}\Big) \\
& + \|x - y\|\Big(|\lambda|\frac{\tau^\beta}{\beta}\Big\{1 + \frac{\tau^\beta}{\beta|\Omega|}\Big(1 + |\mu|\frac{\tau}{\beta+1}\Big)\Big\}\Big) \\
=\ & (\mathcal{L}\Lambda_1 + \Lambda_2)\|x - y\|.
\end{aligned}
$$

Taking the norm of the above inequality for $t \in [0,\tau]]$, we obtain $\|\mathcal{G}(x) - \mathcal{G}(y)\| \leq (\mathcal{L}\Lambda_1 + \Lambda_2)\|x - y\|$, which, in view of the condition (15), implies that the operator $\mathcal{G}$ is a contraction. Hence the operator $\mathcal{G}$ has a unique fixed point by contraction mapping principle, which corresponds to a unique solution of the problem (1). □

Example. Let us consider the following boundary value problem

$$
\begin{cases}
T_{1/4}\Big(T_{3/4} + 1/9\Big)x(t) = f(t,x(t)), \quad t \in J := [0,2], \\
x(0) = 0, \quad x(2) = 1/5 \int_0^2 x(s)ds.
\end{cases}
\tag{16}
$$

Here $\alpha = 1/4$, $\beta = 3/4$, $\lambda = 1/9$, $\mu = 1/5$, $\tau = 2$ and $f(t,x(t))$ will be fixed later.

Using the given data, we find that $|\Omega| \approx 1.729844$, $\Lambda_1 \approx 20.444444$, and $\Lambda_2 \approx 0.645956$, where Ω, Λ_1, and Λ_2 are given by (8), (12) and (13) respectively.
For illustrating Theorem 3 we take

$$f(t,x) = \frac{1}{4\sqrt{900+t}}\Big(\frac{|x(t)|}{|x(t)|+1} + \tan^{-1} x(t) + e^{-t}\Big). \tag{17}$$

Clearly $f(t,x)$ is continuous and satisfies the condition (H_1) with

$$\psi(t) = \frac{\pi + 2(1+e^{-t})}{8\sqrt{900+t}}.$$

Also

$$\Lambda_2 \approx 0.645956 < 1.$$

Thus all the conditions of Theorem 3 are satisfied and consequently there exists at least one solution for the problem (16) with $f(t,x)$ given by (17) on $[0,2]$.
In order to illustrate Theorem 3, we choose

$$f(t,x) = \frac{e^{-t}}{(12+t)^2}\Big(\sin x + \cos t\Big). \tag{18}$$

It easy to check that $f(t,x)$ is continuous and satisfies the Lipschitz condition with $\mathcal{L} = 1/144$. Also

$$\mathcal{L}\Lambda_1 + \Lambda_2 \approx 0.7879314 < 1.$$

Thus the hypothesis of Theorem 3 holds true. Hence the problem (16) with $f(t,x)$ given by (18) has a unique solution on $[0,2]$.

4 Conclusions

We introduced a new form of Langevin equation involving conformable differential operators of different orders and studied it subject to the integral boundary conditions. The existence results for the given problem are derived with the aid of standard fixed point theorems. Our results are new and reduce to the ones for a second-order integral boundary value problem when $\alpha, \beta \to 1$.

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Bashir AHMAD
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203,
Jeddah 21589, Saudi Arabia.
Email: bashirahmad_qau@yahoo.com

Ravi P. AGARWAL
Department of Mathematics,
Texas A& M University,
Kingsville, Texas 78363-8202, USA
Email:Ravi.Agarwal@tamuk.edu

Madeaha ALGHANMI
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203,
Jeddah 21589, Saudi Arabia.
Email: madeaha@hotmail.com

Ahmed ALSAEDI
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University, P.O. Box 80203,
Jeddah 21589, Saudi Arabia.
Email: aalsaedi@hotmail.com